Alternative Default Definitions and the Structure of Credit Spreads

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Abstract

We develop a simple structural credit risk model that allows for two different debt maturities, short-term and long-term debt, and apply it to three alternative default regimes each of which is characterized by its own default boundary: (i) an endogenously determined default boundary, (ii) a liquidity driven default boundary, and (iii) a default boundary in the style of a widely-used external valuation tool that can be categorized as exogenous or principal protective (Moody´sKMV). We derive closed-form valuation formulas for equity, firm and debt value (including credit spreads) and apply them to a base case parameter set. Our main results are: First, for many parameter constellations the liquidity constraint barrier yields valuations considerably deviating from those of the endogenous and the KMV-type barrier. Second, in contrast to many existing models, which face diminishingly small credit spreads for short-term debt, our specification yields significantly positive short-term credit spreads, even for very low leverage levels. And third, we find that the equity holders´ incentive to increase risk in order to raise equity value depends on the type of default boundary to be applied.

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1. Introduction

The current crisis, since its outbreak in 2007, has been demonstrating impressively the necessity to consider alternative default triggers for valuing corporate capital positions. In the course of the crisis so far, we have seen defaults attributable to a variety of reasons such as liquidity shortages, reductions in equity values, strategic decisions by equity holders or non-intervention by third-party stakeholders (potentially other market participants or central governments). Ex ante, however, it is not clear which of the economically and legally motivated default triggers will be the decisive one for the default of a specific firm.

There exist theoretical approaches attempting to deal with the risk attached to default barrier specification.[1] Nonetheless, the question "Which default boundary empirically applies in which situation and for which firm?" remains open by these theoretical considerations. Only recently, Davydenko (2007) identified empirical ex post patterns in the triggers of realized defaults. He differentiates between liquidity shortage and low market asset value. His results show that asset value is the best univariate default predictor, but its information value has to be qualified given that different firms default at very different asset values. Concerning the impact of liquidity shortages on defaults, Davydenko finds that it is significant only when external financing is hard to obtain. The conclusion, he draws from the empirical observations, is that boundary-based default predictions misclassify a large number of firms in cross-section.

Such findings indicate some regime dependence of default barriers. Given the wide cross-sectional variability, however, the influence of regime setting factors on an individual level seems by far not sufficient to rely on exclusively for modeling and quantification purposes. In the context of carrying out valuation tasks, considerable ex ante risks afflicted with default boundary specification remain.

In this paper, we therefore are interested, first, in pricing implications caused by application of alternative default barriers and, second, in debt maturity structure modeling (more precisely, in the introduction of a second finite debt maturity into a structural credit risk model).

[1] Examples in this regard are Giesecke und Goldberg (2004), who combine a first-passage structural credit risk framework with components of incomplete information, presuming that investors cannot observe the default barrier of a firm. The notion of incomplete information in credit risk modeling was introduced by Duffie and Lando (2001), who work with the assumption that investors can observe the true asset value only imperfectly. Another possibility of default specification would be to assume a random but observable default barrier as is done, for example, in Nielsen et al. (1993). Finally, one might perceive the default barrier as an unknown deterministic function that can be extracted backwards from empirical bond data by applying appropriate estimation procedures.

Concerning the first issue, default boundaries, we want to provide some quantification of the risk arising with ex ante default boundary specification. The work is based on a structural credit risk framework, in which we consider the three more or less standard default barrier definitions.[2] These three default categories considered are: principal protective or exogenously driven defaults, endogenous defaults, and liquidity triggered defaults.

Principal protective or exogenously driven defaults can be broadly categorized as defaults triggered when liabilities are exceeding assets[3] or certain modifications of this basic rule are fulfilled. In our work, we will represent this default barrier category by an approximation of the Moody´sKMV[4] default barrier. The latter has been successfully applied in the context of probability of default forecasts over the last couple of years, what can be interpreted as an indication that it captures relevant empirical default characteristics. The Moody´sKMV-type default barrier we apply can be seen as a modification of the criterion liabilities exceeding assets towards consideration of liquidity aspects. Integration of liquidity aspects occurs implicitly as debt is differentiated between short- and long-term, with long-term debt being half-weighted than short-term debt when entering the default barrier.

The endogenously derived default threshold reflects a different perspective.[5] It is obtained by maximizing equity value with respect to the default barrier. Equity holders thereby identify the asset value threshold (lower boundary), at which it makes from their point of view no more difference if the firm continues to exist or goes bankrupt. Application of the endogenous default criterion includes the possibility of overruling the other two criteria, especially that of illiquidity, by the way of external capital injections, if this appears to be a value increasing measure.

The third and last barrier category we consider is the liquidity constraint criterion. The liquidity constraint criterion is intended to capture the risk of illiquidity as one main trigger of

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[2] Contrary to Leland (2004), in this work we do not compare different models with different default definitions but rather integrate three alternative default formulations into one single model and thereby isolate the effects of the default boundary formulation. An example for work that already pursued a similar approach is Leland (1994a), who considers two different default barriers (endogenous and positive net worth covenant) in his structural model. However, he associates the two alternative default definitions with different debt maturities where the first one should typically serve for long-term and the second one for short-term debt.

[3] There is a long list of models that work with a default barrier of such a type, from those we reference here for exemplification purposes only Merton (1974) as a European option quantification and Longstaff and Schwarz (1995) as an American option quantification.

[4] For documentation on the Moody´sKMV credit risk model see Kealhofer (2003a, 2003b). Moody´sKMV models credit risk by applying an extension of Merton (1974).

[5] The number of models applying an endogenous default boundary is large. A prominent representative thereof is the work by Leland (e.g., Leland (1994a) or Leland and Toft (1996)).

bankruptcy. Illiquidity occurs if the internal capacity of the firm to generate cash flows is not sufficient to cover regular payouts, given that no external funds are provided in addition. [6]

The chosen default barrier definitions represent three different perceptions of which type of event will finally trigger the default of a firm. From our perspective, the economically and legally most relevant categories of default triggers are captured by this set.

The second road we pursue in this paper is the one of modeling corporate debt maturity structure. The basis for modeling credit risk in a structural framework is to specify the asset value process of the firm plus the structure of the contingent claims on the underlying asset value. While the asset value process is regularly assumed as geometric Brownian motion and equity is treated as one homogeneous balance sheet position, modeling efforts often focus on the structure of debt positions. Empirically representative corporate capital structures will usually exhibit a number of different debt classes. These debt classes may vary with respect to maturity, seniority, collateral, and possibly further discriminatory characteristics. Modeling efforts aimed at extending the basic structural framework by integrating additional debt characteristics - like consideration of more than one debt maturity, or of seniority and collateral aspects - are important in order to compare results from such models with results from models exhibiting more simplistic debt representations. If results change significantly as soon as a more complex debt structure is introduced, empirical work carried out so far might suffer either from certain valuation deficiencies resulting from too far-reaching reductions in debt structure complexity. Or, it might be afflicted with a bias in sample selection. The bias can result if, as a consequence of simplistic models, only firms with relatively simple capital structures can be considered for the sample set.[7]

To capture the evolution over time of the set of contingent claims, that constitute total debt of a firm, requires a characterization of the dynamics of debt. For this, one can either directly work with the leverage process[8] or one can use a separate representation of asset value (or cash flow respectively) and debt dynamics.[9] Leland and Toft (1996), for example, work with a separate representation of the two and achieve to define a finite maturity debt process (contrary to infinite maturity debt) that fulfills the requirement of time-independence. They

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[6] An example for application of a liquidity type default barrier is the model by Ericsson (2000).

[7] Eom et al. (2004), for example, one of the most comprehensive empirical studies in this field, who implement a selection of different structural models for US corporate bond data, have to restrict themselves to investigating firms with rather simple debt structures. The latter is in line with other empirical studies on this topic.

[8] An example therefore is Dangl and Zechner (2004).

[9] Fischer, Heinkel, and Zechner (1989), Leland (1994a and 1994b), Leland and Toft (1996), and Goldstein, Ju, and Leland (2001) are prominent representatives of a design of such type.

can do so by establishing a stationary debt structure via regular redemption of old and issue of new debt. This leads to a stable level of principal outstanding at every point in time. Leland and Toft (1996) work with single maturity debt (at issuance) and do not consider differences in seniority or collateral.

The contribution of our paper with respect to debt structure modeling is to introduce the possibility of more than one class of finite maturity debt. To obtain this debt differentiation, we extend the framework developed in Leland (1994b) towards inclusion of two different debt maturity classes, short-term and long-term debt.[10] Each of the debt classes will have average finite maturity. Debt structure in Leland (1994b) is characterized by issuance of infinite maturity debt plus redemption and new issuance at a constant rate, thereby a certain average finite maturity of outstanding debt can be generated. The average finite maturity of outstanding debt is determined by the chosen retirement rate. Time-homogeneity is achieved through constant repayment of principal and constant coupon payments. By extending the Leland (1994b) framework to allow for two maturity classes of debt, we hope to bring capital structures used for modeling purposes closer to empirically observed debt structures, which usually comprise more than one debt maturity. We provide analytical solutions for equity, firm, and debt values (including credit spreads) for the case of two different debt classes with different average finite maturities. We investigate the effects a differentiation of the debt structure has on valuation results. This is a prerequisite for assessing the consequences of debt structure simplifications made with respect to maturity categories in credit risk modeling so far, and to gain additional insights if those simplifications are maintainable for the future or should be reconsidered.

The main results of this work are: First, for many parameter constellations, our illustrations demonstrate an astonishingly similar behavior of results based on the endogenous and the KMV-type default barrier. The liquidity constraint, however, often yields considerably different valuation results. Second, our model specification yields significantly positive short-term credit spreads even for very low leverage levels. This is in contrast to comparable existing models, i.e., to models, which also do not include jumps or similar short-term spread increasing components. Third, with respect to equity value we find that only in case the endogenous default barrier applies, an increase in asset volatility also leads to an increase in equity value (for medium to high leverage). Especially when the liquidity constraint applies,

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[10] An extension to multiple debt maturity classes would work analogously to the approach applied here.

such a rebound in equity value is absent. The equity holders´ incentive to increase risk in order to raise equity value therefore depends on the type of default boundary to be applied.

Overall, integration of a second debt maturity into a structural credit risk model seems to have a significant effect on valuation results and helps to accommodate certain empirical findings. Albeit the model remains simple and traceable and can still be easily implemented. It thereby preserves one of its most appealing characteristics. Furthermore, the extent of deviation in valuation results caused by the three different default barriers hints at the magnitude of risk arising from the ex ante uncertainty concerning default barrier specification.

The rest of this paper is organized as follows: In Section 2, we first introduce the model formulation by Leland (1994b), whose stationary debt structure with one class of average finite maturity debt builds the starting point of this work. Then, we present our extension of the Leland (1994b) framework by including an additional class of debt with different average finite maturity. We derive analytical solutions for equity, firm, and debt values (including credit spreads) for each of the three different default criteria. Section 3 carries out a comparative analysis of the analytically obtained results and Section 4 concludes the paper. Complementary illustrations of our valuation results are presented in appendices A to D.

2. The model

Before developing solutions for debt, equity and firm value plus credit spreads we need to define the three essential model components. These components are the asset value process and the debt structure of the firm as well as the default barrier in its three different specifications.

Asset Dynamics: The asset value of the firm at time t is denoted by $A(t)$ and follows a geometric Brownian motion. Asset value dynamics are given by:

$$dA(t) = (\mu - \beta)A(t)dt + \sigma A(t)dZ(t)$$

where $dZ(t)$ is a standard Brownian motion, μ is the gross growth rate, β the payout rate, and $\mu - \beta$ the (net) growth rate of the assets. Volatility σ is assumed to be constant.

2.1 Multiple finite maturity debt classes

The debt instruments considered in this paper and for which closed-form valuation results will be presented are coupon paying bonds with arbitrary average maturity.

Debt Structure: We use a stationary debt structure of the type developed in Leland (1994b). This debt structure is characterized by issuance of infinite maturity debt plus introduction of a constant redemption and new issuance rate m, by which a certain average finite maturity of outstanding debt can be generated. This means that a constant fraction m of outstanding principal P, mP, is continuously retired and replaced by newly issued debt of the same principal, coupon, and seniority. Therefore, at each point in time the firm has debt with constant total principal P, paying a constant total coupon rate C. Debt is issued at market value and retired at principal value. Although the new debt is assumed to have the same coupon and principal, the price at which debt can be sold depends on current asset value A. The average finite maturity of outstanding debt is determined by the chosen retirement rate m: The higher the retirement rate m, the shorter will be the average maturity of debt. Time-homogeneity is achieved by constant repayment of principal and constant coupon payments. Leland (1994b) refers to sinking-fund provisions as real-world equivalent to this payment scheme.

It is assumed that a constant fraction of the currently outstanding principal of each tranche is amortized per unit of time and per tranche. The advantage of this type of sinking fund provision over other types, like one where a constant fraction of initial rather than remaining principal is retired each moment, is that under the chosen one units of debt issued at different times are identical (in the sense of debt characteristics and debt value per unit) at time t. This, however, would not be the case for the second specification mentioned above (redemption of a constant fraction of initial principal), what in turn would imply that each tranche had to be valued separately. Leland and Toft (1996), in their approach to model finite maturity debt for one single class of debt worked with a debt structure of the second type. Their firm continuously issues debt principal at a specified rate, which has maturity T at the time of issuance. Simultaneously it retires the currently maturing principal amount from previously issued debt tranches. Contrary to Leland (1994b), units of different outstanding tranches (i.e., of different maturities) need not have the same market value and therefore each of the tranches has to be valued separately. This increases complexity of the calculations. Leland and Toft (1996) state that the formulas for bond prices obtained in Leland (1994b) are relatively simple but still preserve much of the behavior observed with the model presented in Leland and Toft (1996).

The type of debt structure chosen for our paper therefore brings about analytical convenience as it causes relatively moderate modeling efforts while it still allows pursuing one of the main purposes of this work, namely introducing more than one class of average finite maturity debt and analyzing the effects thereof. In the following, we extend the Leland (1994b) debt structure by integrating an additional class of debt with different average finite maturity.

We differentiate between two debt classes: short-term debt (denoted by *S*) and long-term debt (denoted by *L*). Total outstanding principal of short-term debt is denoted by P_S, total outstanding principal of long-term debt by P_L. Short-term debt pays a constant total coupon rate of C_S and long-term term debt one of C_L. At every instant of time a fraction m_S (m_L) of P_S (P_L), $m_S P_S$ ($m_L P_L$), is retired and replaced by newly issued debt of the same principal and coupon. $m_S P_S$ ($m_L P_L$) therefore defines p_S and p_L, the continuously retired and newly issued principal of short-term and long-term debt. With $p_i(\tau,t)$ we denote the principal of debt issued at time $\tau \leq t$ and still outstanding at time *t*, where *i* = *S, L*. $p_S(\tau,t)$ and $p_L(\tau,t)$ follow the subsequent differential equation:

$$\frac{\partial p_i(\tau,t)}{\partial t} = -m_i p_i(\tau,t) .$$

These dynamics imply that the outstanding principal of debt issued at time τ declines exponentially with time as can be seen from the following equation.

$$p_i(\tau,t) = p_i e^{-m_i(t-\tau)}$$

where $p_i = p_i(\tau, \tau)$. The same relationship holds for the coupon to be paid on debt of each tranche: $c_i(\tau,t) = c_i e^{-m_i(t-\tau)}$, where again $c_i = c_i(\tau,\tau)$.

We then, when $m_i > 0$, get for P_i (the total outstanding principal of short-term debt or long-term debt respectively):

$$P_i = \int_{-\infty}^{t} p_i(\tau,t)d\tau = \frac{p_i}{m_i}.$$

The coupon for short-term or long-term debt respectively, when $m_i > 0$, is given by

$$C_i = \int_{-\infty}^{t} c_i(\tau,t)d\tau = \frac{c_i}{m_i}.$$

Given the described debt structure, the average maturity of each class of debt results in the following way:

$$M_i = \int_{0}^{\infty} t m_i e^{-m_i t} dt = \frac{1}{m_i}.$$

We assume that the retirement rate for short-term debt is higher than that for long-term debt ($m_S > m_L$). This implies that the average maturity of short-term debt is smaller than the average maturity of long-term debt ($M_S < M_L$). [11]

Given that no bankruptcy occurs, the cash flows to debt holders (coupon plus principal retirement) due to currently issued debt, at a future time *t*, result as $(c_i + m_i p_i)e^{-m_i t}$.

In case of bankruptcy, which will occur if the asset value of the firm strikes the lower boundary A_D, a fraction α of the asset value is lost and the remainder $(1-\alpha)A_D$ is distributed to debt holders. This allocation rule reflects the absolute priority assumption, where in case of bankruptcy debt holders receive all the remaining asset value while equity holders get nothing. From the amount remaining to be distributed to debt holders, a fraction x_S goes to

[11] Within one debt category (i.e., within a debt class associated with the same average maturity M_i) all tranches pay the same coupon and all currently outstanding debt will be amortized at the same rate m_i. Therefore, all outstanding units of debt from one debt category will be assigned identical market values. The only difference between older and younger tranches will be that there will be fewer units of older tranches outstanding than of younger ones.

holders of short-term debt and a fraction x_L to holders of long-term debt (with $x_S = \frac{P_S}{P_S + P_L}$ and $x_L = \frac{P_L}{P_S + P_L}$).

Following Leland´s (1994b) derivations and adapting them to the two-debt-classes-scenario, which we focus on in this paper, leads us as an intermediate step to $d_i(0)$. $d_i(0)$ is the value of currently issued debt in each debt category i (short-term and long-term).

$$d_i(0) = \frac{c_i + m_i p_i}{r + m_i}\left[1 - \int_{t=0}^{\infty} e^{-(r+m_i)t} f(t; A, A_D)dt\right] + x_i m_i (1-\alpha) A_D \left[\int_{t=0}^{\infty} e^{-(r+m_i)t} f(t; A, A_D)dt\right]$$

with: $\int_{t=0}^{\infty} e^{-(r+m_i)t} f(t; A, A_D)dt = \left(\frac{A}{A_D}\right)^{-\gamma_i}$

where r is the risk-free rate, $f(t; A, A_D)$ is the transition probability (i.e., the density of the first passage time of a Brownian motion to touch a certain barrier from above),

and $\gamma_i = \frac{(r - \beta - 0.5\sigma^2) + \left[(r - \beta - 0.5\sigma^2)^2 + 2(m_i + r)\sigma^2\right]^{1/2}}{\sigma^2}$.

Market value of debt issued at time τ, with $\tau \leq 0$, and currently (at time 0) still outstanding, is $d_i(0)e^{m_i \tau}$.[12] Total value of outstanding debt per maturity category i, D_i, is obtained by integrating over $-\infty \leq \tau \leq 0$, which gives $D_i = \frac{d_i(0)}{m_i}$. So, when we divide the equation for $d_i(0)$ by m_i and apply the relationships $P_i = \frac{p_i}{m_i}$ and $C_i = \frac{c_i}{m_i}$, the following values for short-term and long-term debt result.

Value of Short-Term Debt: According to the derivations made above, the value of short-term debt is given by

$$D_S = \frac{C_S + m_S P_S}{r + m_S}\left[1 - \left(\frac{A}{A_D}\right)^{-\gamma_S}\right] + x_S (1-\alpha) A_D \left(\frac{A}{A_D}\right)^{-\gamma_S}$$

where γ_S is represented by the following term

[12] All outstanding units of debt have the same price. The only difference between older and newer tranches is that there are fewer units of older tranches outstanding.

$$\gamma_S = \frac{\left(r-\beta-0.5\sigma^2\right)+\sqrt{\left(r-\beta-0.5\sigma^2\right)^2+2\left(m_s+r\right)\sigma^2}}{\sigma^2}.$$

Value of Long-Term Debt: Analogously to short-term debt, the value of long-term debt results as

$$D_L = \frac{C_L+m_L P_L}{r+m_L}\left[1-\left(\frac{A}{A_D}\right)^{-\gamma_L}\right]+x_L(1-\alpha)A_D\left(\frac{A}{A_D}\right)^{-\gamma_L}$$

with

$$\gamma_L = \frac{\left(r-\beta-0.5\sigma^2\right)+\sqrt{\left(r-\beta-0.5\sigma^2\right)^2+2\left(m_L+r\right)\sigma^2}}{\sigma^2}.$$

The Firm Value: The firm value is given by the asset value (A) plus the value of the tax benefits (TB, where τ_T is the corporate tax rate) minus the value of the bankruptcy costs (BC). The values for tax benefits and bankruptcy costs are represented by

$$TB = \frac{\tau_T(C_S+C_L)}{r}\left[1-\left(\frac{A}{A_D}\right)^{-\gamma_0}\right] \qquad BC = \alpha A_D\left(\frac{A}{A_D}\right)^{-\gamma_0}$$

Then the firm value results as

$$v(A) = A+\left(\frac{\tau_T(C_S+C_L)}{r}\right)\left[1-\left(\frac{A}{A_D}\right)^{-\gamma_0}\right]-\alpha A_D\left(\frac{A}{A_D}\right)^{-\gamma_0}$$

with

$$\gamma_0 = \frac{\left(r-\beta-0.5\sigma^2\right)+\sqrt{\left(r-\beta-0.5\sigma^2\right)^2+2r\sigma^2}}{\sigma^2}.$$

Following Leland (1994b) we would like to emphasize that the above formulation of tax consideration assumes that there are tax benefits whenever the firm is not in bankruptcy. If a loss of tax deductibility below some threshold $A_T > A_D$ were considered, this would raise A_D and thereby lower the values for equity and debt.[13]

[13] Further note, that γ_0 (given by the equation for γ_i with $m_i = 0$) and not γ_i is the exponent here. The reason therefore is that total coupon C (C_S+C_L) and total principal P (P_S+P_L) are constant. Total tax benefits and bankruptcy costs are not being reduced over time, contrary to the outstanding amount of principal of each tranche of debt.

The Equity Value: The equity value of the firm is given by the total firm value minus the value of short-term debt and the value of long-term debt

$$E(A) = v(A) - D_S(A) - D_L(A) .$$

Credit Spreads: Calculation of credit spreads is done in this paper in the following way:

$$CS_i = \frac{C_i}{D_i} - r \qquad \text{with } i = S, L.$$

2.2 Alternative default boundaries

The subsequently derived results for our three alternative default boundary specifications (endogenous, liquidity constraint, and KMV-type) can be used for obtaining closed-form solutions for debt, equity, and firm value.

Endogenously determined default barrier: We start with the payout rate to stockholders, which is given by the term $(\beta A - C_S - C_L - p_S - p_L + d_S + d_L)$. This payout rate declines as A falls and may also become negative. In such a case new equity has to be issued to meet the payment requirements for bonds. Bankruptcy will occur in the endogenous case then, when the firm has reached an asset value level A_D where equity value is no longer sufficient to cover required coupon payments plus refunding costs. Bankruptcy is triggered when the asset value has fallen to a level where equity can no longer be issued to meet the debt service requirements. As the required amount of debt service is infinitesimal over an interval *dt*, equity value is *0* when asset value falls to A_D: $E(A_D) = 0$

We now can derive the endogenous bankruptcy criterion by maximizing the value of equity with respect to the default boundary:

$$\left.\frac{\partial E(A)}{\partial A}\right|_{A=A_D} = 0 .$$

We get the following relationship:

$$0 = 1 + \left[\frac{\tau_T (C_S + C_L)}{r}\right]\gamma_0 \frac{1}{A_D} + \alpha\gamma_0 - \left[\frac{C_S + m_S P_S}{r + m_S}\right]\gamma_S \frac{1}{A_D} + x_s(1-\alpha)\gamma_S -$$

$$- \left[\frac{C_L + m_L P_L}{r + m_L}\right]\gamma_L \frac{1}{A_D} + x_L(1-\alpha)\gamma_L .$$

The asset value level that triggers endogenously determined default is then given by:

$$A_D^E = \frac{\left[\frac{C_S + m_S P_S}{r + m_S}\right]\gamma_S + \left[\frac{C_L + m_L P_L}{r + m_L}\right]\gamma_L - \left[\frac{\tau_T (C_S + C_L)}{r}\right]\gamma_0}{1 + \alpha\gamma_0 + x_S(1-\alpha)\gamma_S + x_L(1-\alpha)\gamma_L}.$$

The asset value A_D^E represents the endogenously determined default threshold.

Liquidity constraint: We now turn to a default threshold arising from a strictly interpreted liquidity constraint. The liquidity constraint criterion is intended to capture the risk of illiquidity as one main trigger of bankruptcy. Illiquidity occurs if the internal capacity of the firm to generate cash flows is not sufficient to cover regular payouts, given that no external funds are provided in addition. We refer in the context of this section to the illiquidity concept used in Ericsson (2000).

For a firm to stay solvent the following condition has to hold:

$$\beta A(t)dt + d\big(A(t)\big) \geq (1-\tau_T)\big(C_S + C_L\big)dt + m_S P_S dt + m_L P_L dt.$$

In the above stated liquidity constraint, we have on the left-hand side (which comprises the regular cash inflows) with $\beta A(t)dt$ the operating cash flows generated by the firm and with $d\big(A(t)\big)dt$ the proceeds from newly issued debt. On the right-hand side (representing the cash outflows), the term $(1-\tau_T)\big(C_S + C_L\big)dt$ reflects the coupon payments (after consideration of tax benefits) while $m_S P_S dt + m_L P_L dt$ captures the principal repayments. Also here we assume that tax benefits can be realized whenever the firm is solvent. If a loss of tax deductibility below some threshold $A_T > A_D$ were considered, this again would raise A_D.

Debt issued (short-term and long-term) at the time default occurs recovers the following amount, taking into consideration that in case of default the amount $(1-\alpha)A_D$ is distributed to bond holders:

$$x_S(1-\alpha)m_S A_D + x_L(1-\alpha)m_L A_D.$$

Therefore we get as limiting scenario, which defines the liquidity constraint default boundary:

$$\beta A_D + (1-\alpha)A_D\big(x_S m_S + x_L m_L\big) = (1-\tau_T)(C_S + C_L) + m_S P_S + m_L P_L.$$

The liquidity constraint default barrier we finally obtain is given by the following expression:

$$A_D^L = \frac{(1-\tau_T)(C_S + C_L) + m_S P_S + m_L P_L}{\beta + (1-\alpha)(x_S m_S + x_L m_L)}.$$

KMV-type default barrier: The third and last barrier category we consider will be an approximation of the Moody´sKMV[14] default barrier. The Moody´sKMV default barrier has been successfully applied in the context of probability of default forecasts over the last couple of years. This can be interpreted as an indication that it captures relevant empirical default characteristics. We have to work with an approximation of the proprietary Moody´sKMV default barrier definition, however, because to our knowledge there has no unambiguous definition authorized by Moody´sKMV been around.

This approximation, which we call a Moody´sKMV-type default barrier, can be seen as a modification of the criterion liabilities exceeding assets towards consideration of liquidity aspects. Integration of liquidity aspects occurs implicitly as debt is differentiated between short- and long-term, with long-term debt being half-weighted than short-term debt when entering the default barrier.

The KMV-type default barrier definition, which we use as an approximation of the proprietary default boundary applied by Moody´sKMV, is the following:

$$A_D^{KMV} = P_S + 0.5P_L \,.$$

From this equation it can be seen that applying A_D^{KMV} for valuation purposes triggers bankruptcy when the asset value of the firm falls to a minimum level equivalent to the total principal of short-term debt plus half of the principal of long-term debt currently outstanding.

[14] For documentation on the Moody´sKMV credit risk model see Kealhofer (2003a, 2003b). Moody´sKMV models credit risk by applying an extension of Merton (1974).

3. Comparative analysis

In this section we want to give some illustrations of our theoretical results. These applications will consider all three default boundaries and will be grouped into the following three categories: 1. credit spread analysis with respect to leverage and asset volatility, 2. credit spread analysis with respect to debt maturity structure and long-term debt maturity, and 3. evaluation of equity value behavior. Complementary results on firm value, equity values as function of asset value, default boundaries, and effects of maturity differentiation are available in appendices A to D. The first step of the following analysis will be to discuss the parameter selection process leading us to the base case parameter set, which will establish the starting point for our numerical evaluations.

3.1 Parameter selection

Risk-free interest rate r: We use US Treasury yields to derive a base case scenario for the risk-free interest rate. The table below displays US Treasury yields provided by the Federal Reserve Board. The rates are calculated as historical averages (10Year: 1993-2002, 20Years: 1983-2002) of weekly observations on Treasury yields for 1, 5, and 10 years to maturity.

Treasury yields			
	1Year Treasury Yield	*5Year Treasury Yield*	*10Year Treasury Yield*
10Year Average	4.76%	5.59%	5.89%
20Year Average	6.18%	7.16%	7.50%

Table 1
Source: The Federal Reserve Board, Treasury Constant Maturity Series

Our point of reference concerning maturity will be the 5Year Treasury yield (based on the 10Year average estimate). The reason for opting for this maturity is the model characteristic of a constant risk-free rate with a flat term structure. As we are going to integrate short-term and long-term claims into our analysis, the model implementation will require some compromise in the form of intermediate-term maturity. The risk-free interest rate we use in our base case scenario is 5.5%.

Bankruptcy costs α: Most of the empirical information available on loss data refers to recovery rates. Altman and Kishore (1996) and rating agencies such as Moody´s and Standard & Poor´s provided broad empirical information on recovery rates. In the following, two brief examples of such data are given.

Standard & Poor´s: Average recoveries						
	All	*Bank debt*	*Senior secured notes*	*Senior notes*	*Senior subor-dinated notes*	*Junior subor-dinated notes*
Mean rec. rate	51.14%	83.54%	63.75%	49.92%	28.18%	12.81%

Table 2
Source: Standard & Poor´s (2001), p.82, Table 1

Altman and Kishore: Weighted-average recovery rates					
	Senior secured	*Senior unsecured*	*Senior subordinated*	*Subordinated*	*Zero coupon*
Average rec. rate	57.89%	47.65%	34.38%	31.34%	21.66%

Table 3
Source: Altman and Kishore (1996), p.58, Table 1

In the present context we have to work with bankruptcy costs, however, and not with recovery rates. The relationship between recovery rates (RR) and bankruptcy costs (α) is given by RR = $[(1-\alpha)*A_D] / P$ (where P is the face value of debt). But as for two of the three default boundaries we use in this paper (namely the endogenous default barrier and the liquidity constraint) the default boundary A_D is dependent on α, we cannot simply back out α from the appropriate recovery rate. We rather have to concentrate on empirical data for bankruptcy costs directly. Andrade and Kaplan (1998), for example, estimate costs of financial distress in the range of 10 to 20 percent of the firm value. In line with this result and with Huang and Huang (2003), who find the figure consistent with a recovery rate of around 51 percent for senior unsecured bonds (see also the figures for senior notes and senior unsecured in the tables above, 49.92 percent and 47.65 percent respectively), we decided to use bankruptcy costs α equal to 15 percent in the base case scenario.

Payout ratio β: Huang and Huang (2003) state that β, in principal, should be the average total net annual payments to holders of all of the firm´s securities - expressed as a fraction of the firm´s asset value. As a rough estimate of the payout ratio they use the leverage-weighted sum of the average historical coupon rate and the average dividend yields. They approximate β with 6 percent: 0.65 * 4 percent + 0.35 * 9 percent = 5.75 percent, where 4 percent is the average dividend yield, 9 percent the average coupon rate, and 0.35 the average S&P 500 leverage. Eom et al. (2004), in contrast, work with single bond data and therefore can also consider share repurchases in their payment ratio. The sample statistics on all firms included in their analysis exhibits an average payout ratio of 4.83 percent.

Given the 4.83 percent in Eom et al. (2004), which - from a conceptual point of view - can be seen as a bit more comprehensive, and, on the other hand, the 5.75 percent from Huang and Huang (2003), we decided to use an average payout ratio of 5 percent.

Asset volatility σ: Crosbie and Bohn (2003, figure 4) present a categorization of asset volatility estimates according to different groups of industries and asset sizes. In this categorization estimates range for low risk industries and large asset sizes from below 5 percent up to 35 percent for high risk industries in combination with relatively small asset sizes. Eom et al. (2004), in their sample summary statistics, state an average asset volatility of 23.6 percent.

As we do neither have a special focus on the high risk nor on the low risk end of firms when we formulate the base case scenario for our model illustrations, our point of reference will be the average asset volatility of 23.6 percent by Eom et al. (2004). For the base case scenario we therefore work with an asset volatility of 24 percent, which is also close to Leland (2004), who applies 23 percent.

Remaining input parameters: The tax rate, which is relevant for tax deductibility of interest payments, is set at 35 percent. This is in line with the model implementations in Eom et al. (2004) and Leland and Toft (1996). Coupon rates for long-term and short-term debt are set at 7 percent and 6 percent respectively. A numerical identification of a unique long-term and short-term coupon pair such that market value of debt equals face value is not possible in the setting of this work, as market value of debt is dependent on three different default barriers. The latter characteristic would lead us to three different coupon pairs instead of one. Base case leverage is set at 50 percent (but a comprehensive sensitivity analysis will be carried out and presented anyway). The proportion of long-term to short-term debt is fixed with 60 percent to 40 percent in order to reflect a slight overhang of long-term debt financing. Sensitivity towards these figures will be evaluated. Average maturity of long-term and short-term debt will be in the base case scenario 10 years and 1.25 years respectively. Again, we consider variations of these maturities to get a notion of how sensitive results are with respect to average maturities. The chosen allocation rule between short-term and long-term debt in case of bankruptcy is tied to the proportion of short-term versus long-term debt principal, that is, short- and long-term debt are equally treated in case of bankruptcy. Furthermore, we do not introduce different seniorities for different debt categories, what could be achieved via adapting the allocation rule.

The base case parameter set					
Principal of short-term debt	P_S	40	Fraction of bankruptcy proceeds distributed to short-term debt $[P_S/(P_S+P_L)]$	x_S	0.4
Principal of long-term debt	P_L	60	Fraction of bankruptcy proceeds distributed to long-term debt $[P_L/(P_S+P_L)]$	x_L	0.6
Coupon short-term debt (P_S*0.06)	C_S	2.4	Asset starting value	A_0	200
Coupon long-term debt (P_S*0.07)	C_L	4.2	Risk-free rate	r	0.055
Fraction of short-term debt retired	m_S	0.8	Payout rate	β	0.05
Fraction of long-term debt retired	m_L	0.1	Asset volatility	σ	0.24
Average maturity short-term debt ($1/m_S$)	M_S	1.25	Tax rate	τ_T	0.35
Average maturity long-term debt ($1/m_L$)	M_L	10	Leverage (principal / asset value)	L	0.5
Fraction lost in bankruptcy	α	0.15			

Table 4

3.2 Credit spread analysis with respect to leverage and asset volatility

Given the inverse relationship between credit spread and debt value (with credit spread equivalent to coupon divided by debt value minus risk-free interest rate) we will discuss, without loss of information, only the results for credit spreads. These results will cover long-term as well as short-term credit spreads.

The first step of the analysis is to investigate the behavior of long-term credit spreads for varying leverage levels and an elsewise base case parameter set. We see from figure 1 that increasing leverage levels lead to rising long-term credit spreads for all of the three default criteria. The leverage induced spread difference is 126 basis points at a maximum (observable for the endogenous case).

For very low leverage levels (i.e. in our case, 5 percent) all of the three model formulations start at the same credit spread of 88 basis points. Up to a leverage of 35 percent, results for the three default criteria either are the same or at least lie in a very narrow range. Endogenous and KMV-type results lie in a very narrow range (of maximal 3 basis points) even up to leverage levels of 70 percent. For very high leverage levels (in our case, 90 percent) credit spread estimates are 2.14, 2.06, and 1.81 percent for the endogenous, the KMV-type, and the liquidity constraint model respectively.

The endogenous default boundary yields the highest credit spreads over all leverage levels, while those of the liquidity constraint barrier are the lowest ones (reflecting the high values of the liquidity constraint default barrier and thereby low risk of suffering losses on part of the bond holders).

Figure 1 spread estimates for higher leverage levels (about 50 percent upwards) are lower than those obtained in Leland and Toft (1996, figure 4). But the opposite is true for lower leverage levels (about 40 percent downwards), despite the high bankruptcy costs of 50 percent used in their work. Overall, in our model setting there seems to be less spread sensitivity with respect to pure changes in leverage. Also Collin-Dufresne and Goldstein (2001, e.g. figure 3) obtained less influence of initial leverage on spread levels in their approach applying stationary leverage. Their spread estimates for highly levered firms are considerably lower when stationary leverage ratios, and not constant debt amounts, are used for modeling purposes.

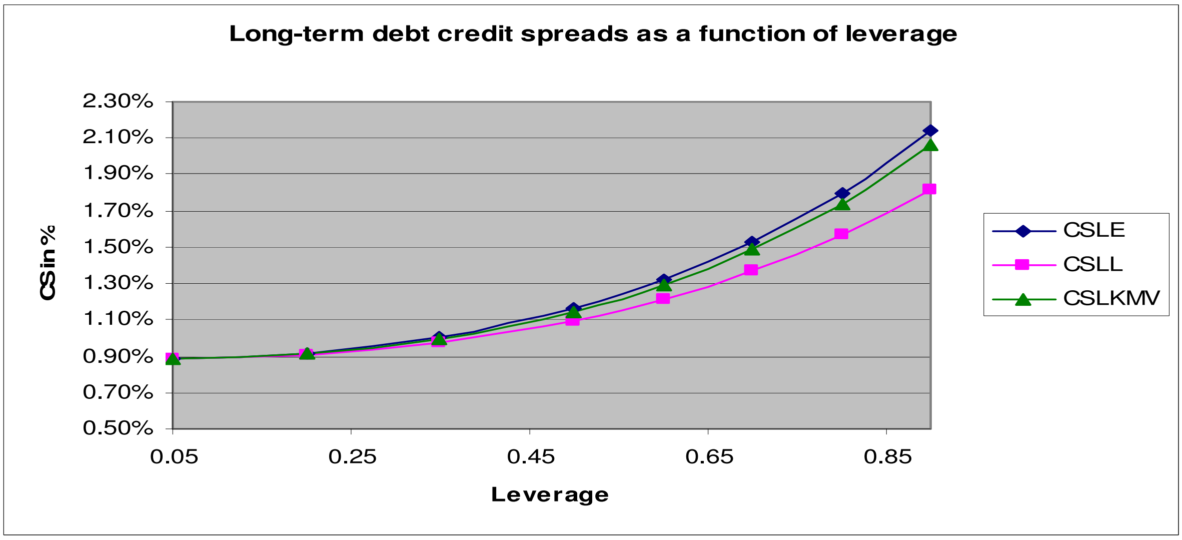

Figure 1

In the second step we analyze the behavior of long-term credit spreads with respect to simultaneous variation of leverage and asset volatility (with the rest of the parameter specifications again according to the base case parameter set). The results show that endogenous and KMV-type default barriers yield similar long-term credit spreads with respect to levels as well as sensitivity shapes.[15] The model version using the liquidity constraint barrier (see figure 3), on the contrary, is less sensitive to combined increases in leverage and asset volatility than the other two. The highest long-term credit spread estimates from the liquidity constraint model (for the combination high leverage / low volatility) lie at 1.88 percent while those from the two others rise to more than 3 percent (for the combination high leverage / high volatility). The effect of volatility on long-term credit spreads can be

[15] Therefore only the endogenous model results are depicted in figure 2.

considered as substantial – especially for endogenous and KMV-type model formulations and especially for high leverage levels.

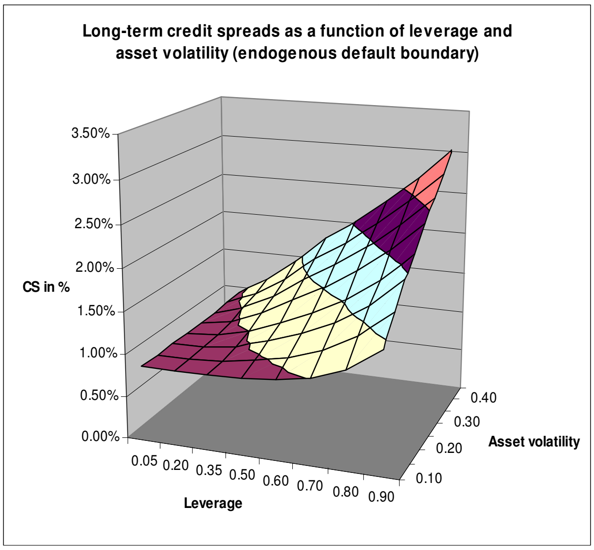

Figure 2

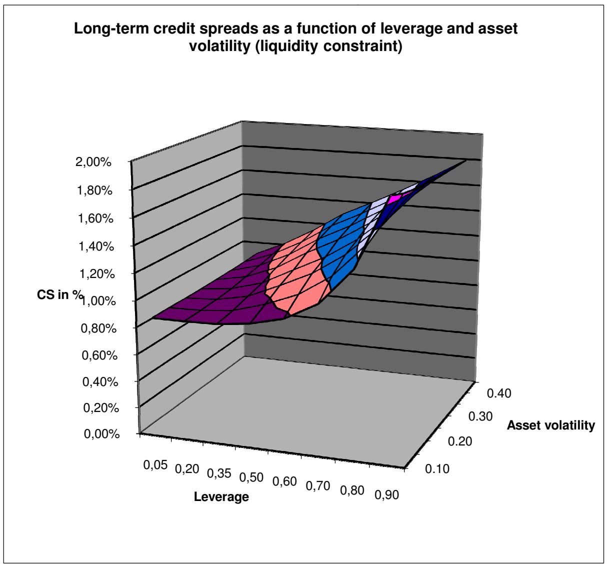

Figure 3

If we turn now to short-term credit spreads, for all three default barriers credit spreads start to increase only at or above a leverage ratio of 50 percent (see figure 4). Below this level, short-term credit spreads do not exhibit sensitivity with respect to leverage increases. Despite the increases from 50 percent onwards, short-term credit spreads for high leverage levels are still much lower than those obtained in Leland and Toft (1996, figure 3) for the same maturity as our average short-term maturity (1.25 years).[16]

For medium to low (and even for very low) leverage levels, our model formulations do yield positive short-term credit spreads of somewhat below 50 basis points under the base case scenario. Overall, also for short-term credit spreads there seems to be less sensitivity with respect to pure leverage changes in our model setting than in comparable established models. For low to medium leverage levels, Leland and Toft´s (1996) model formulation, for instance, does not yield credit spreads different from zero for maturities of up to more than one year (in case of longer overall firm debt maturities, even up to more than two years). Also Collin-Dufresne and Goldstein (2001, figures 2 and 4) exhibit zero credit spreads at the short end of the term structure (up to about 5 years in their parameter setting), though obtaining a spread increase for low initial leverage levels (e.g., 15 percent) in the area of medium to longer debt

[16] We consider an average short-term debt maturity of 1.25 years as representative maturity for a corporate capital structure and therefore present no further differentiations for even shorter average maturities of short-term debt.

maturities. This characteristic is independent of whether they consider stochastic interest rates or not, as stochastic interest rates primarily affect longer-term maturities.

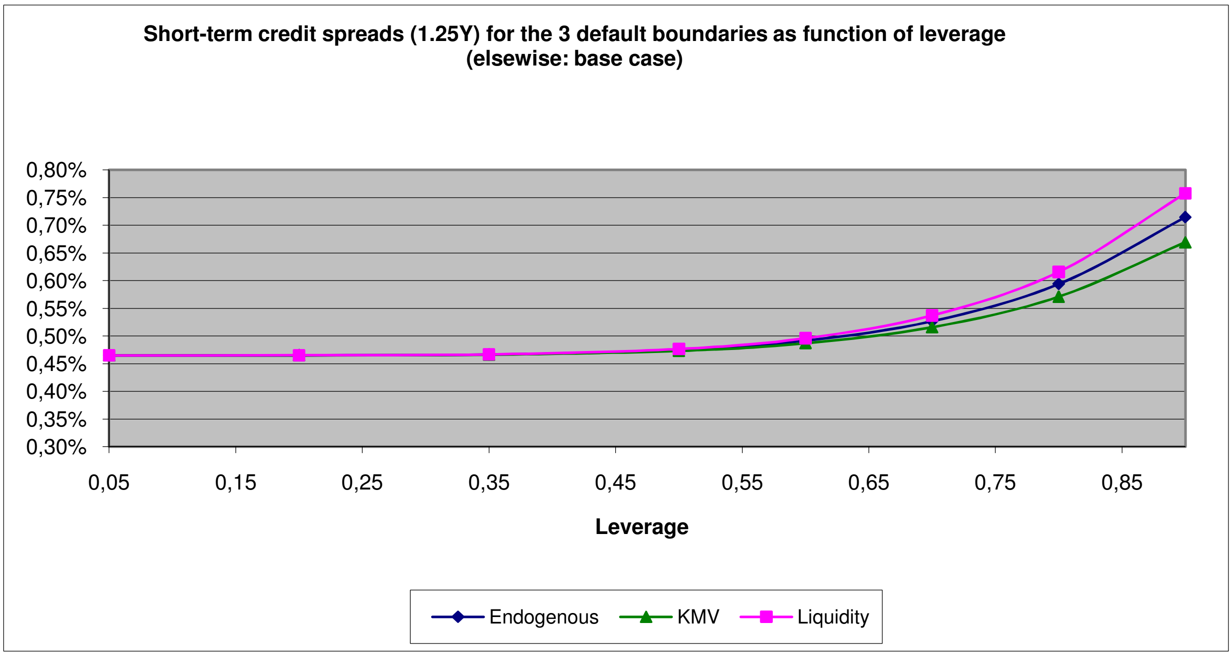

Figure 4

Short-term spread sensitivity with regard to asset volatility is low. One can observe, e.g., an increase from 47 basis points for 10 percent asset volatility to 56 basis points for 45 percent asset volatility for the endogenous case, whose results are representative also for the other two default barriers. For the endogenous and the KMV-type model, both leverage and asset volatility have to be high for significant increases in short-term credit spreads. Estimates for short-term credit spreads range from 0.47 percent (low leverage, low volatility) to 1.02 percent (high leverage, high volatility) for the endogenous model and from 0.47 to 1.20 percent for the KMV-type model. In the liquidity constraint model, short-term credit spread sensitivity towards leverage and volatility is observable mainly with respect to the first. Short-term credit spreads vary from 0.47 percent (low leverage, low volatility) to 0.81 percent (high leverage, low volatility).

3.3 Credit spread analysis concerning maturity structure and long-term debt maturity

The third dimension of analysis is long-term credit spread sensitivity with regard to changes in debt maturity structure (i.e., short-term versus long-term debt proportion of the capital structure). Debt maturity structure changes cause only very small effects on long-term credit spreads under the base case parameter set. The bandwidth of debt maturity structure variations we consider in our calculations ranges from 0.11 up to 4.00 (these figures represent the ratio of nominal short-term debt to nominal long-term debt). Endogenous and KMV-type models show a debt maturity structure induced increase in long-term credit spreads from 1.12 percent to 1.19 percent and 1.07 percent to 1.20 percent respectively (in each case for an increase in the short-term debt proportion from 0.11 to 4.00). Therefore, in the endogenous and the KMV-type model there is a small but still observable effect of more short-term debt implying higher credit spreads. Long-term spreads in the liquidity constraint model, however, fall from 1.12 percent to 1.08 percent when the short-term debt proportion increases.

The fourth direction towards which long-term credit spreads are evaluated is their dependence on long-term debt maturity. This analysis does not result in the formulation of a term structure in a narrower sense (i.e., credit spreads for different debt maturities of a specific firm) but rather in the specification of a spread curve for average long-term debt of firms with different average long-term debt maturities. As one can infer from the valuation results depicted in figure 5, for all of the three default barrier specifications long-term credit spreads fall with increasing long-term debt maturity.[17] This pattern reflects the decreasing refinancing risk associated with longer-term debt maturities. Leland and Toft (1996) basically also observe this relationship for medium to high leverage levels (apart from the short end of the maturity structure), but contrary to us not for lower leverage levels (of, e.g., 40 percent).

Long-term credit spreads (for the base case leverage of 50 percent) are for the endogenous and the KMV-type default boundary very similar over all long-term debt maturities. Spreads span an interval from around 1.30 percent (for an average long-term debt maturity of 3 years) to 0.96 percent (for 35 years). Hence the debt maturity induced difference in long-term credit spreads lies slightly above 30 basis points. Long-term credit spreads in the liquidity constraint specification are for shorter debt maturities also approximately the same as in the two others, but they exhibit slightly more sensitivity towards increases in long-term debt maturity. In the liquidity constraint case, credit spreads for an average long-term debt maturity of 35 years are 0.88 percent, the debt maturity induced difference between credit spreads for 3 years and 35 years is therefore about 40 basis points.

One can furthermore see from figure 6 that in our model formulations a maturity reduction leads to a decrease in spread deviations between different leverage levels. Figure 6 shows long-term credit spreads as a function of average long-term debt maturity for high, medium,

[17] There are only minor exceptions from this relationship: For the endogenous (see figure 6) and the KMV-type model versions, at the shorter end of the maturity structure one can observe minimal spread increases of a few basis points in case of very high leverage levels (e.g., 70 percent).

and low leverage (on the basis of the endogenous default boundary).[18] When we now move along the curves from, for instance, 15 to, e.g., 7 years of average long-term debt maturity, we see the distance between the three curves shrinking. The contrary is true for Leland and Toft (1996, figure 4): The lower maturities become, the more pronounced is the effect of rather large spread increases caused by already rather small leverage increases (except at the very short end).

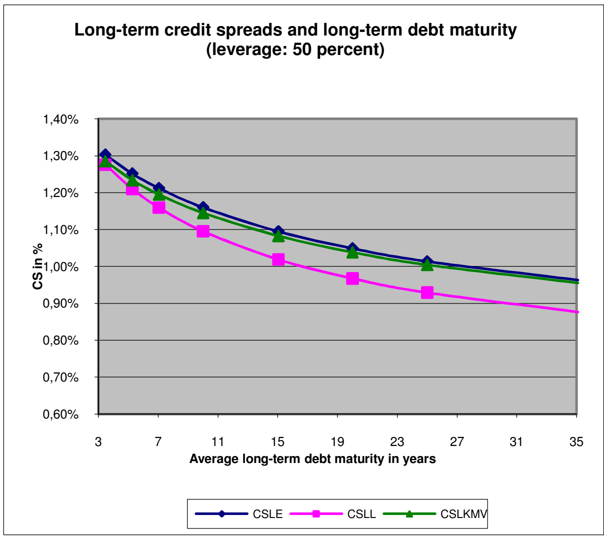

Figure 5

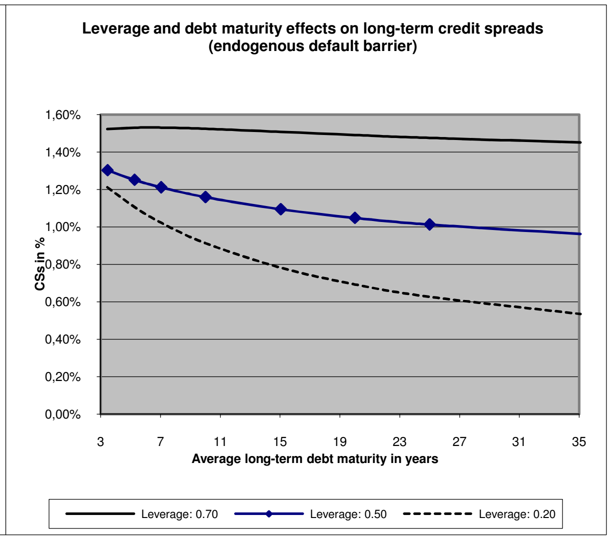

Figure 6

On short-term credit spreads, debt maturity structure and long-term debt maturity have de facto no effect. This holds for all of the three model specifications. The lack of influence of average long-term debt maturity on short-term credit spreads persists even if we additionally take into account leverage as a second dimension (see table 5). The opposite holds for the results in Leland and Toft (1996, figure 3). They find for their term structure analysis of different bond maturities that short-term spreads for the same leverage levels are considerably dependent on the debt maturity of the firm, especially for high leverage levels. For instance, their 1-year-spread pertaining to a leverage level of 70 percent and a firm debt maturity of 20 years amounts to about 1 percent, while the 1-year-spread for a firm debt maturity of 5 years arises to 2.5 percent.[19]

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[18] We depict here only the case of the endogenous default barrier as the other two barrier formulations yield rather similar shapes. Results for them are available on request.

[19] In case of low to medium (50 percent) leverage levels, the primary effects of different firm debt maturities in Leland and Toft (1996) are different periods of zero spreads at the short end of the term structure. For a firm debt maturity of 5 years, e.g., the zero spread period for short-term debt is somewhat more than 1 year, while it is about 2.5 years for a firm debt maturity of 20 years.

Short-term credit spreads and their dependence on average long-term debt maturity (low, medium, and high leverage)									
Leverage	***20%***			***50%***			***70%***		
Average long-term debt maturity in years	***5Y***	***10Y***	***20Y***	***5Y***	***10Y***	***20Y***	***5Y***	***10Y***	***20Y***
Short-term credit spreads (endogenous)	0.47%	0.47%	0.47%	0.48%	0.47%	0.47%	0.53%	0.53%	0.52%
Short-term credit spreads (liquidity)	0.47%	0.47%	0.47%	0.48%	0.48%	0.48%	0.53%	0.54%	0.54%
Short-term credit spreads (KMV-type)	0.47%	0.47%	0.47%	0.47%	0.47%	0.47%	0.52%	0.52%	0.52%

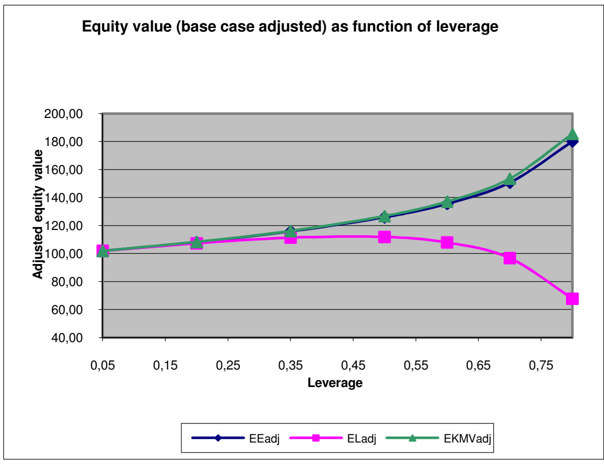
Table 5

3.4 Equity value

Concerning equity value dependence on leverage we find the following behavior of our model specifications for an elsewise base case scenario: For low leverage levels (up to 20 to 30 percent) all of the three default boundaries produce similar results for adjusted equity value.[20] The higher leverage turns, the larger becomes the difference between the endogenous / KMV-type pair at one hand and the liquidity constraint version at the other hand. For the endogenous and the KMV-type case, adjusted equity values are increasing with rising leverage levels, while for the liquidity constraint version equity value does so only up to a medium leverage level of about 50 percent. When leverage rises further, adjusted equity values start falling in the liquidity constraint case (figure 7). It therefore also exhibits the lowest maximum equity value of the three barrier types.

For low leverage levels, also firm values are very similar for all three default barriers (figure 8). Firm value reaches its maximum for the endogenous and the KMV-type case at leverage ratios between 60 and 70 percent, while in the liquidity constraint case it peaks at around 35 percent leverage (for an elsewise base case scenario). For high leverage levels, the liquidity constraint version exhibits the strongest negative sensitivity towards rising leverage and is consequently associated with the lowest firm value for high leverage as well as the lowest maximum firm value out of the three model formulations. Further results on firm value characteristics are available in appendix A (paralleling the structure of the results on equity value presented in the rest of this chapter).

[20] When we present equity values for varying leverage levels, we use base case adjusted equity values. To obtain the adjusted values we take the model results for equity values and adjust them through multiplication by the relationship “base case equity ratio to current equity ratio”. Equity ratios, in this context, are given as “one minus principal short-term plus principal long-term debt divided by the asset value”.



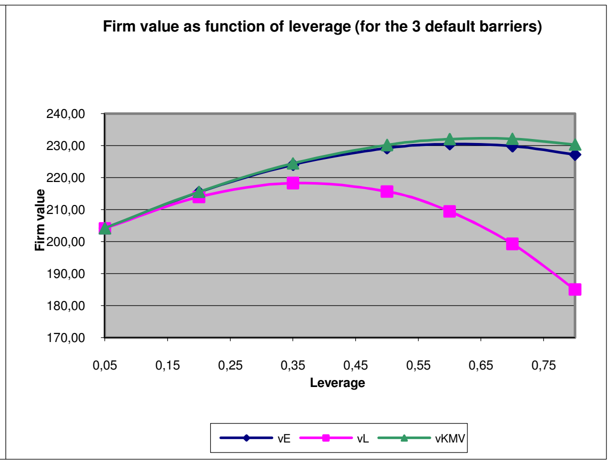


Figure 7 *Figure 8*

When expanding the analysis towards a second dimension, the dimension of volatility, we find the following characteristics in the behavior of adjusted equity value (table 6). In the endogenous default barrier case (figure 9), adjusted equity value increases with leverage for all volatility levels. For low leverage levels, adjusted equity value is a decreasing function of volatility, while from medium (around 50 percent) leverage upwards dependence of equity on volatility starts becoming U-shaped: the higher leverage rises, the lower are volatility levels at which the equity curve turns. The lowest adjusted equity value in our setting results for the combination low leverage / high volatility while the highest one arises for high leverage accompanied by high volatility.

Also for the KMV-type case, adjusted equity values are increasing with leverage for all volatility levels analyzed. Furthermore, equity values are decreasing with rising volatility levels. There is, however, one minor exception of this last observation at the very high end of leverage, where one can identify some small rebound of equity when asset volatility rises towards 40 percent and higher. But a distinct U-shape as for the endogenous default boundary results cannot be observed. The lowest adjusted equity value in the KMV-type version results again for the combination low leverage / high volatility while the highest one arises for high leverage / low volatility.

In the liquidity constraint setting, adjusted equity value is again a decreasing function of volatility for all leverage levels (figure 10). With respect to optimal leverage there is for each volatility level a leverage ratio that maximizes equity value. The higher asset volatility becomes the lower is this equity maximizing leverage. The highest adjusted equity value under the liquidity constraint results for low asset volatility and above medium leverage (60 percent). Across the three model versions, the liquidity constraint yields in our setting by far the lowest maximum equity value while the KMV-type barrier produces the highest.

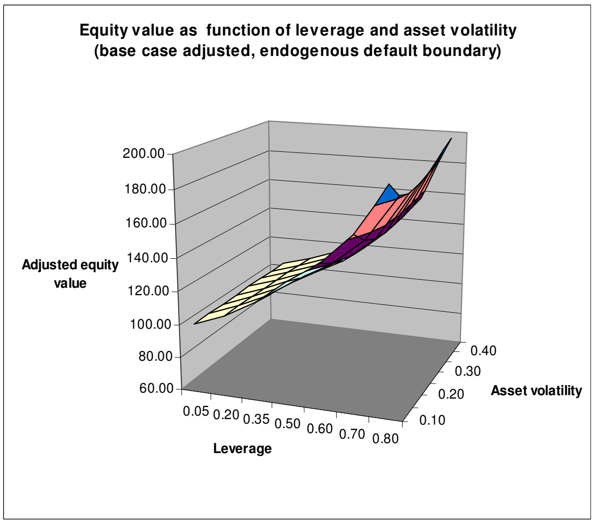


Figure 9

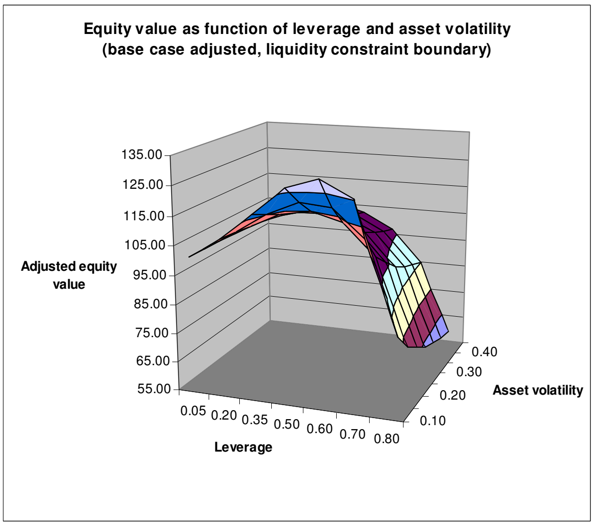


Figure 10

Adjusted equity value							
Endogenous default barrier				*Leverage*			
Volatility	*0.05*	*0.20*	*0.35*	*0.50*	*0.60*	*0.70*	*0.80*
0.10	101.89	108.95	118.83	132.85	145.68	163.38	**192.10**
0.15	101.89	108.76	117.71	129.54	140.12	155.20	**181.84**
0.20	101.88	108.46	116.58	127.17	136.93	151.62	**179.40**
0.24	101.87	108.20	115.84	125.93	135.57	150.63	**180.10**
0.30	101.84	107.84	115.03	124.89	134.79	150.91	**183.34**
0.35	101.81	107.59	114.59	124.55	134.88	152.08	**187.10**
0.40	101.78	107.38	114.31	124.51	135.38	153.72	**191.32**
0.45	101.74	107.22	114.15	124.70	136.16	155.66	**195.78**
Liquidity constraint				*Leverage*			
Volatility	*0.05*	*0.20*	*0.35*	*0.50*	*0.60*	*0.70*	*0.80*
0.10	101.89	108.88	117.91	127.32	**130.69**	125.27	93.50
0.15	101.89	108.45	115.43	**120.22**	119.29	109.94	78.93
0.20	101.87	107.81	113.00	**114.96**	111.98	101.27	71.48
0.24	101.84	107.26	111.30	**111.78**	107.88	96.69	67.72
0.30	101.78	106.44	**109.18**	108.17	103.45	91.93	63.92
0.35	101.72	105.82	**107.73**	105.87	100.73	89.09	61.70
0.40	101.65	105.23	**106.47**	103.99	98.55	86.86	59.98
0.45	101.57	104.69	**105.37**	102.38	96.72	85.01	58.56
KMV-type default barrier				*Leverage*			
Volatility	*0.05*	*0.20*	*0.35*	*0.50*	*0.60*	*0.70*	*0.80*
0.10	101.89	108.97	119.10	134.43	149.87	173.73	**217.95**
0.15	101.89	108.83	118.21	131.48	144.29	163.84	**200.33**
0.20	101.88	108.54	117.04	128.63	139.77	157.02	**190.11**
0.24	101.87	108.26	116.13	126.77	137.12	153.47	**185.53**
0.30	101.84	107.82	114.96	124.69	134.44	150.29	**182.19**
0.35	101.80	107.47	114.16	123.47	133.03	148.91	**181.31**
0.40	101.76	107.15	113.51	122.59	132.13	148.23	**181.37**
0.45	101.72	106.85	112.97	121.94	131.57	147.98	**181.95**

Table 6. Adjusted equity value as function of leverage and asset volatility: The dark-grey fields mark the volatility levels yielding the highest equity value per leverage level while the light grey ones identify those resulting in the lowest equity values. The bold values are the highest equity values across leverage levels (as considered in our parameter setting).

Interesting to note on the relationship of equity value, leverage, and asset volatility is the observation drawn from table 6 that only in case the endogenous default barrier applies an increase in volatility also leads to an increase in equity value (for medium leverage levels upwards). Especially if the liquidity constraint boundary applies such a rebound of equity value for higher leverage and higher volatility is absent, and with a minor exception this holds for the KMV-type barrier as well. Consequently, the existence of the potential to achieve higher equity values by increasing volatility depends on the applicability of the type of default boundary.[21]

From the discussion of our results so far has become clear that in a considerable number of cases not the endogenous default boundary but the KMV-type boundary yielded the highest equity value. Given that the formulation of the endogenous default boundary results from maximizing equity value, this observation might be irritating. The reason for the observed pattern is the following. Equity value is a decreasing function of the default barrier: the lower the default barrier, the higher the according equity value. And as for several parameter constellations, the KMV-type boundary specification we apply produces a lower default barrier than the endogenous specification, the KMV-type boundary can also yield higher equity values. A default barrier lower than the endogenous one is rendered possible by the fact that the KMV-type barrier does not fulfill all of the construction requirements that apply for the endogenous barrier. According to Leland (1994b), the conditions the endogenous barrier has to comply with are, first, $E(A_D) = 0$ and, second, the smooth-pasting condition $\left.\frac{\partial E(A, A_D)}{\partial A}\right|_{A=A_D} = 0$. While the first requirement holds for all of our three default barriers, the second does not. It is valid only for the endogenous barrier. For a further discussion of this issue please refer to appendix B.

In the last step, we now will turn to the dependence of equity value on debt maturity structure and long-term debt maturity (for an elsewise base case scenario). For equity an increasing proportion of short-term debt versus long-term debt[22] leads to value decreases when the endogenous or the KMV-type default barrier apply. Equity results for the liquidity constraint barrier are rather insensitive to shifts in debt maturity structure, on the contrary (figure 11).

[21] Leland and Toft (1996, figure 8), who consider an endogenous default barrier, find that the potential for raising equity value by increasing asset volatility is dependent on debt maturity. The longer debt maturity the larger will be the interval of leverage levels where a rise in equity value as consequence of volatility increases would occur – and also the more pronounced will be the effects of such increases.

[22] It is this proportion we refer to as debt maturity structure.

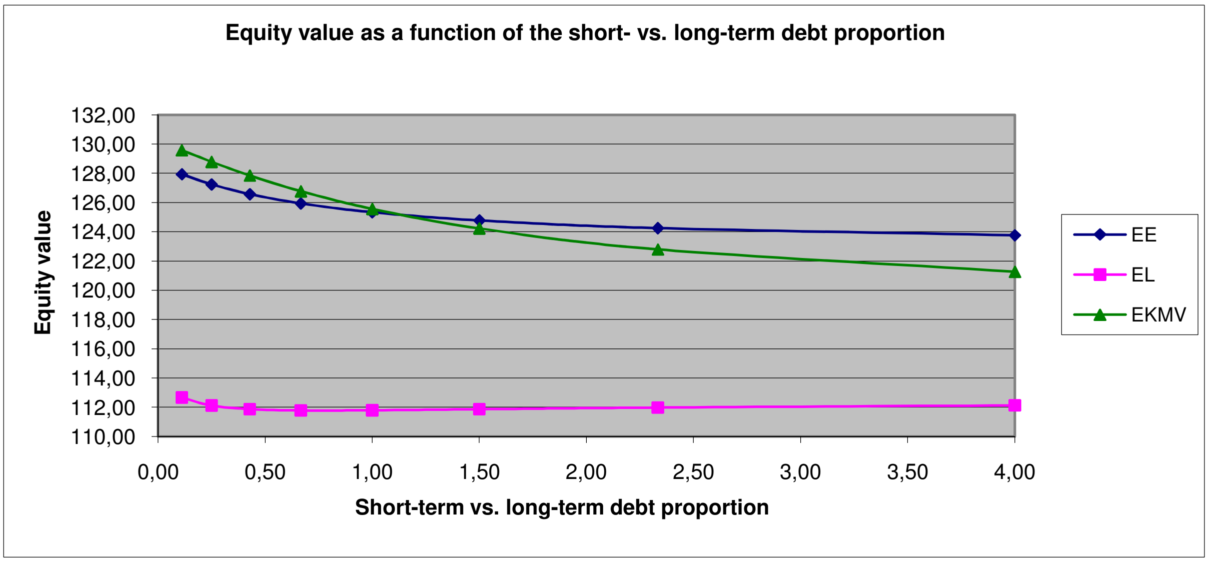


Figure 11

For rising average maturity of long-term debt we cannot identify an increasing effect on equity value (table 7), rather the opposite is true.

Equity values for varying average long-term debt maturity								
	Average long-term debt maturity in years							
Default barriers	*3*	*5*	*7*	*10*	*15*	*20*	*25*	*35*
Endogenous	126.03	126.14	126.11	125.93	125.59	125.28	125.02	124.61
KMV-type	128.08	127.61	127.25	126.77	126.18	125.74	125.41	124.92
Liquidity constraint	113.12	112.68	112.31	111.78	111.12	110.65	110.29	109.79

Table 7. Equity values for varying average long-term debt maturity: This table displays the results for equity values obtained on the basis of the base case parameter set when average long-term debt maturity varies between 3 and 35 years.

4. Conclusion

Corporate capital structure usually exhibits more than one class of debt maturity. We therefore develop a simple structural credit risk model that allows for two different debt maturities. We do so by extending the Leland (1994b) model towards integration of a second debt dimension. Furthermore, we consider three alternative default boundaries (endogenous, liquidity, and KMV-type) in order to admit a representative bandwidth of possible default triggers.

Our results show that for many parameter constellations there are only small valuation differences between the endogenous and the KMV-type default barrier. The liquidity constraint, however, often yields considerably different valuation results. Furthermore, in contrast to many existing models, our specification yields significantly positive short-term credit spreads even for very low leverage levels. Finally, we find that the equity holders´ incentive to increase risk in order to raise equity value depends on the type of default boundary to be applied. Basically, it seems to be limited to the endogenous default barrier.

Overall, integration of a second debt maturity into a structural credit risk model seems to have a significant effect on valuation results and helps to accommodate certain empirical findings. Despite this extension, the model remains simple and traceable and can still be easily implemented. It thereby preserves one of its most appealing characteristics.

The extent of deviation in valuation results caused by the three different default barriers hints at the magnitude of risk arising from the ex ante uncertainty of which barrier will be the actual default trigger in a specific case. The integration of a (financial and / or macroeconomic) state dependent default barrier into a structural framework might be a helpful – though not a sufficient – step for future work.

Appendix A: Firm value

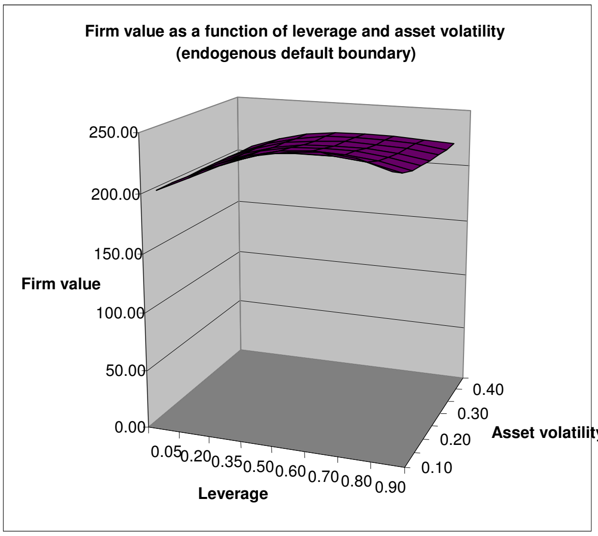
The highest firm values for all leverage scenarios analyzed are obtained for the lowest volatility levels (table A2). This is true for all three types of default barriers. The maximum firm value for the liquidity constraint version occurs in our setting at a medium leverage level, for the endogenous default case at a high leverage level of about 70 percent, and for the KMV-type barrier at the highest leverage analyzed in our examples (80 percent). On the other hand, the lowest firm values for the endogenous and the KMV-type case are obtained at the lowest analyzed leverage levels, while that for the liquidity constraint is obtained for the highest leverage. Overall, the liquidity constraint barrier is the model version with the lowest maximum firm value while the KMV-type model produces the highest one out of the three model versions. Contrary to equity value we can in our setting not identify potential for increasing firm value by raising asset volatility for any of the three default boundaries. Maximum firm value seems to be a question of leverage and not of volatility.

With respect to debt maturity structure, one can observe firm value reductions accompanying increases in short-term debt portions for all of the three default barriers (table A1). The sensitivity towards higher short-term debt portions is in the base case setting most pronounced for the KMV-type barrier, while it is of least importance under the liquidity constraint boundary.

Firm values for varying debt maturity structure								
	Short-term debt to long-term debt amount							
Default barriers	*0.11*	*0.25*	*0.43*	*0.67*	*1.00*	*1.50*	*2.33*	*4.00*
Endogenous	233.18	231.73	230.40	229.17	228.02	226.94	225.93	224.97
KMV-type	235.55	233.80	232.00	230.16	228.28	226.35	224.39	222.40
Liquidity constraint	217.86	216.93	216.22	215.62	215.10	214.62	214.17	213.74

Table A1. Firm values for varying debt maturity structure: This table displays the results for firm value obtained on the basis of the base case parameter set when the proportion of short-term debt to long-term debt varies between 0.11 and 4.00.

For rising average maturity of long-term debt, on the other hand, we cannot identify an unambiguously increasing effect on firm value. Only the endogenous default boundary produces a very small increase of two units for an increase of average long-term debt maturity from 3 to 35 years. Otherwise, firm values remain unaffected. The lack of firm value sensitivity to rising average debt maturity in our model formulations does not parallel the results obtained by Leland and Toft (1996, figure 1), who find a stronger positive influence of increasing debt maturity on firm value for an endogenous default boundary (results for other default barriers are not covered by their study).



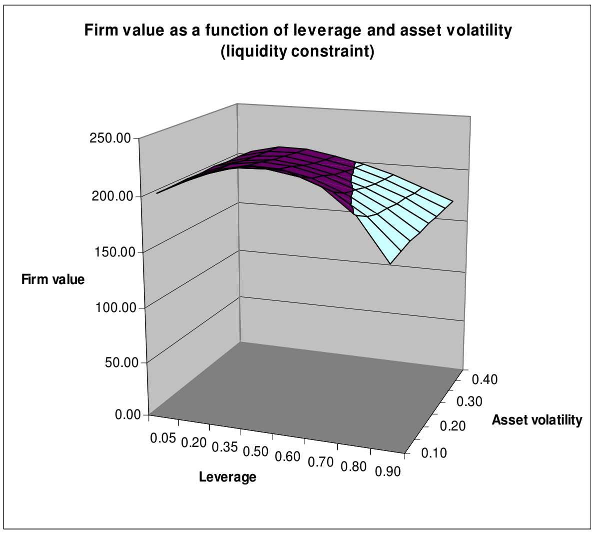


Figure A1

Figure A2

Firm value							
Endogenous default barrier				*Leverage*			
Volatility	*0.05*	*0.20*	*0.35*	*0.50*	*0.60*	*0.70*	*0.80*
0.10	204.20	216.74	228.69	238.69	243.13	**244.66**	242.04
0.15	204.20	216.42	227.09	234.72	237.30	**237.28**	234.11
0.20	204.18	215.89	225.31	231.35	**233.02**	232.51	229.57
0.24	204.15	215.40	223.98	229.17	**230.45**	229.84	227.20
0.30	204.09	214.65	222.24	226.57	**227.55**	226.97	224.77
0.35	204.03	214.05	220.98	224.84	**225.71**	225.21	223.35
0.40	203.95	213.47	219.87	223.38	**224.18**	223.78	222.20
0.45	203.87	212.90	218.85	222.08	**222.84**	222.54	221.21
Liquidity constraint				*Leverage*			
Volatility	*0.05*	*0.20*	*0.35*	*0.50*	*0.60*	*0.70*	*0.80*
0.10	204.20	216.63	227.47	**233.02**	230.67	220.42	198.89
0.15	204.19	215.93	224.10	**225.28**	220.32	209.37	191.20
0.20	204.15	214.86	**220.70**	219.33	213.41	202.84	187.11
0.24	204.10	213.92	**218.27**	215.62	209.41	199.28	184.99
0.30	203.98	212.53	**215.15**	211.31	204.96	195.49	182.82
0.35	203.86	211.42	**212.96**	208.49	202.17	193.19	181.54
0.40	203.72	210.38	**211.04**	206.14	199.90	191.35	180.54
0.45	203.57	**209.39**	209.33	204.12	197.99	189.83	179.72
KMV-type default barrier				*Leverage*			
Volatility	*0.05*	*0.20*	*0.35*	*0.50*	*0.60*	*0.70*	*0.80*
0.10	204.20	216.77	229.05	240.39	246.85	251.90	**254.92**
0.15	204.20	216.53	227.79	236.91	241.24	243.74	**244.04**
0.20	204.18	216.04	225.97	233.05	235.80	**236.71**	235.58
0.24	204.15	215.51	224.40	230.16	232.00	**232.08**	230.30
0.30	204.09	214.62	222.12	226.33	**227.20**	226.48	224.12
0.35	204.01	213.84	220.35	223.56	**223.87**	222.70	220.07
0.40	203.92	213.06	218.70	**221.10**	220.97	219.50	216.70
0.45	203.82	212.28	217.14	**218.87**	218.40	216.69	213.78

Table A2. Firm value as function of leverage and asset volatility: The dark-grey fields mark the volatility levels yielding the highest equity value per leverage level while the light grey ones identify those resulting in the lowest equity values. The bold values are the highest equity values across leverage levels (as considered in our parameter setting).

Appendix B: Equity values as function of asset value

A default barrier lower than the endogenous one is made possible by the fact that the KMV-type barrier does not fulfill all of the construction requirements that apply for the endogenous barrier. According to Leland (1994b), the conditions the endogenous barrier has to comply with are, first, $E(A_D) = 0$ and, second, the smooth-pasting condition $\left.\frac{\partial E(A, A_D)}{\partial A}\right|_{A=A_D} = 0$.

While one can see from figure B1 that the first requirement holds for all of the three default barriers, the second does not. It is valid only for the endogenous default barrier. In figure B1 the KMV-type equity curve once intersects with the x-axis at a higher asset value than the asset value where the endogenous equity curve is tangent to the x-axis and it intersects a second time at a lower asset value than the endogenous barrier. KMV-type equity value becomes negative between the first and the second point of intersection and thereby violates the requirement of limited liability of equity ($E(A) \geq 0$ for all $A \geq A_D$). Irrespective of these theoretical problems, KMV-type barriers are regularly applied in financial industry. This is the reason why we consider it important to integrate at least one specification thereof in our work.

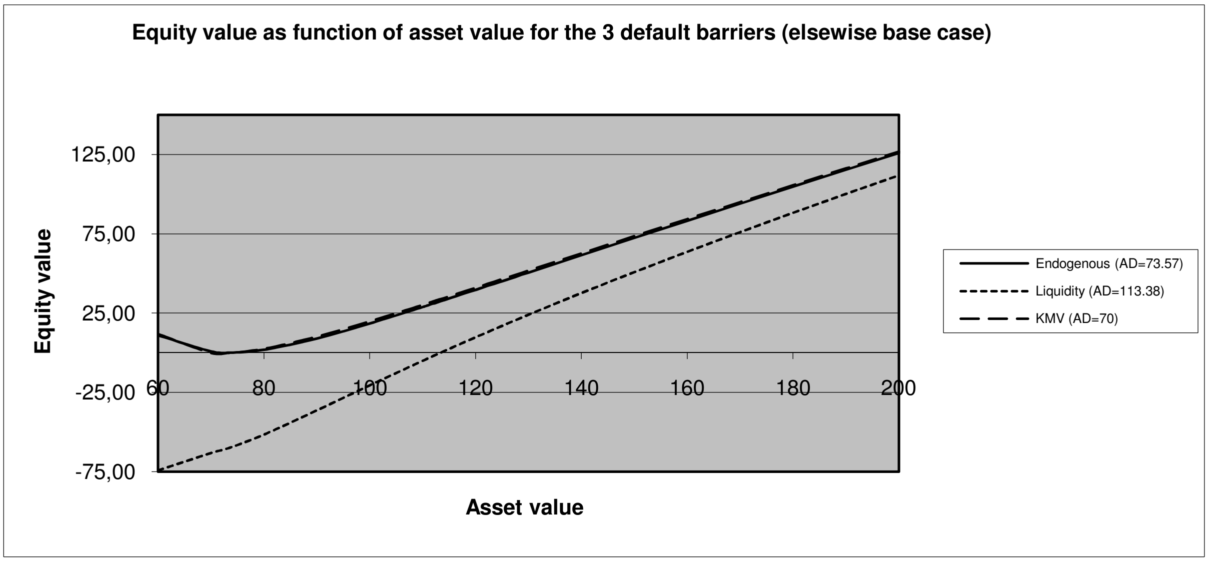


Figure B1

Appendix C: Default boundaries

The shape of default boundaries themselves, which can be considered an intermediate result in the valuation process, is depicted in figures C1 and C2 as function of leverage and asset volatility.[23] The highest default boundary can be observed in each of the three model versions at the highest analyzed leverage level (i.e., 90 percent of an asset value of 200). The liquidity constraint yields the overall highest default barrier for scenarios evaluated in the current setting. This value is independent of volatility and amounts to 204 (thereby it lies above nominal debt). The endogenous boundary follows with already considerably lower 154 (for the combination high leverage / low volatility), while the also volatility independent KMV-type boundary produces the lowest maximum value of 126. The lowest boundary values, on the other hand, occur for all of the three model versions at the lowest leverage level analyzed (i.e., 5 percent of an asset value of 200). While the according value for the endogenous barrier lies at 6 (for the combination low leverage / high volatility), the KMV-type barrier follows at 7, and the liquidity constraint concludes at 11 (what is again above nominal debt).

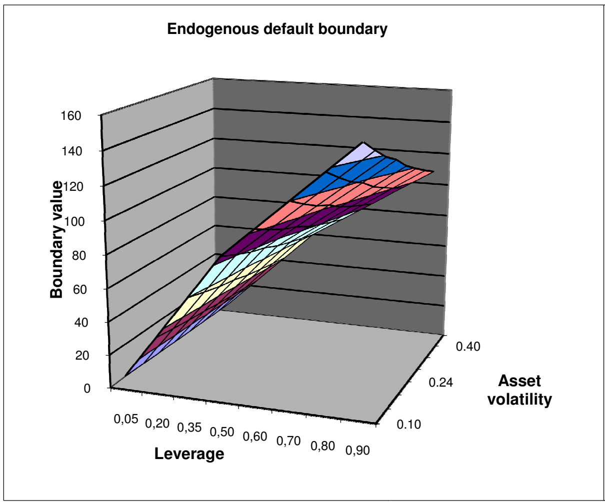


Figure C1. Endogenous default boundary: This graph displays the endogenous default boundary as function of leverage and asset volatility (with the rest of the parameters according to the base case set).

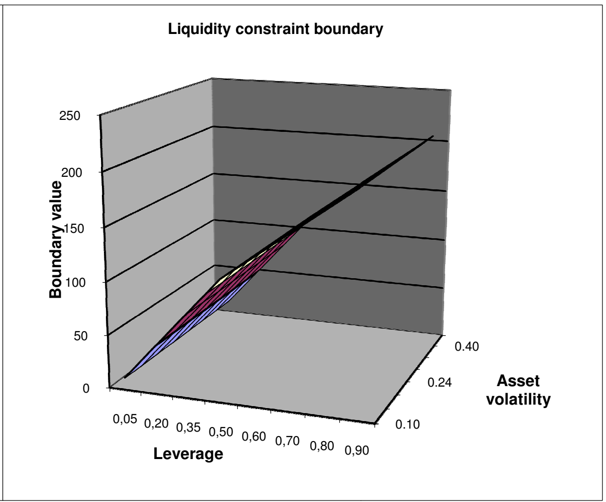


Figure C2. Liquidity constraint boundary: This graph displays the liquidity constraint default boundary as function of leverage and asset volatility (with the rest of the parameters according to the base case set).

Concerning debt maturity structure, the highest default boundary sensitivity to increases in the short-term debt portion is observable for the KMV-type barrier (given an otherwise base case scenario). This is a consequence of the fact that in the KMV-type barrier debt is shifted from

[23] We do not include an extra graph for the KMV-type default barrier, as values therefore are by definition the sum of weighted debt components.

the half-weighted to the full-weighted component when long-term debt is replaced by short-term debt (figure C3).

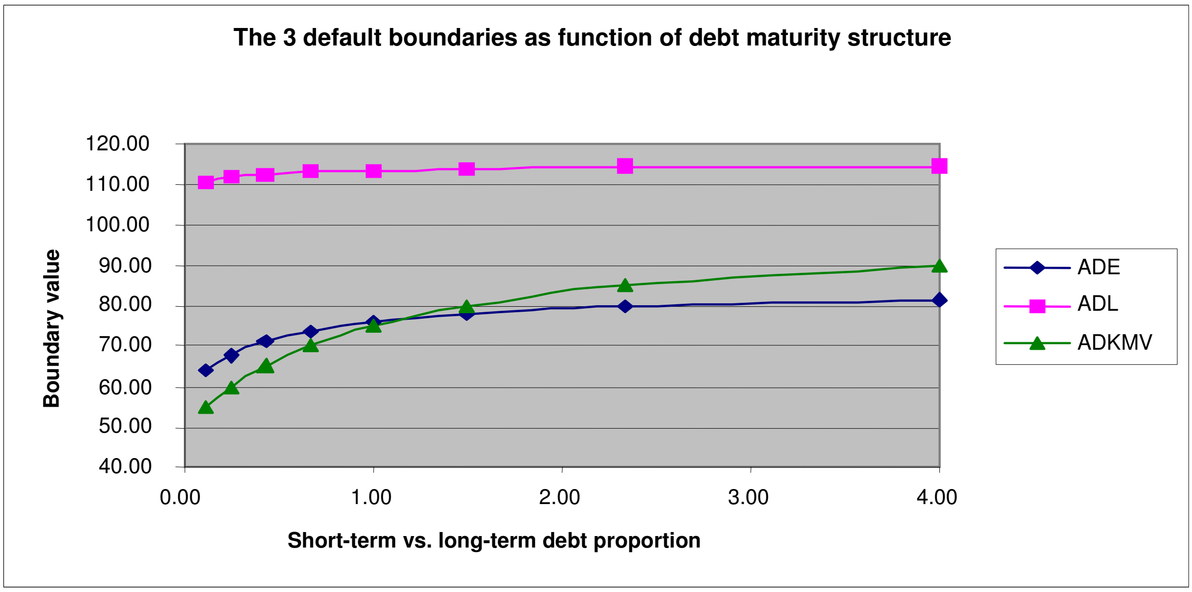


Figure C3. The 3 default boundaries as function of debt maturity structure: The graph displays the results for the three default boundaries evaluated in this work on the basis of the base case parameter set when the proportion of short-term debt to long-term debt varies between 0.11 and 4.00.

Increasing average long-term debt maturity leads for the base case setting to decreasing default boundaries - except for the KMV-type case, where long-term debt maturity has no influence on boundary values. Thereby, the endogenous and the liquidity constraint barrier are reflecting the lower refinancing risk associated with longer-term debt.

Default boundaries for varying average long-term debt maturity								
	Average long-term debt maturity in years							
Default barriers	*3*	*5*	*7*	*10*	*15*	*20*	*25*	*35*
Endogenous	77.89	75.86	74.70	73.57	72.58	72.07	71.76	71.41
KMV-type	70.00	70.00	70.00	70.00	70.00	70.00	70.00	70.00
Liquidity constraint	114.26	113.85	113.61	113.38	113.17	113.06	113.00	112.92

Table C1. Default boundaries for varying average long-term debt maturity: Results in this table are based on the base case parameter set and a variation of average long-term debt maturity from 3 to 35 years.

Appendix D: Effects of maturity differentiation

Selected effects of introducing a second maturity class into the Leland (1994b) valuation framework are shown by the results in table D1, for all of the three default barriers. Assuming that the real firm to be valued is one with a short-term and a long-term debt component we compare the output from our two-maturities model version with that from the one-maturity model by Leland (1994b). Results for the Leland (1994b) model are obtained by applying the weighted average debt maturity to the sum of short- and long-term debt. The extent of output deviations between the two model formulations of up to 4 percent (yet for the base case parameter scenario) may be a further argument for the relevance of debt maturity differentiation in valuation models.

While short-term debt as a consequence of undifferentiated debt maturity would be regularly overvalued, the contrary holds for long-term debt. With respect to firm and equity value the one-maturity results throughout overstate the according two-maturities results by one to four percent. An analysis of credit spread differences between the two model formulations is not appropriate in this context, as it would compare spreads referring to different points in the term structure.

<table>
<tr><th colspan="10">Two maturity classes versus one</th></tr>
<tr><th rowspan="2">Result categories</th><th colspan="3">Endogenous</th><th colspan="3">Liquidity constraint</th><th colspan="3">KMV-type</th></tr>
<tr><th>2 maturities (0.4: 1.25Y 0.6: 10Y)</th><th>Average maturity</th><th>Diff. in %</th><th>2 maturities (0.4: 1.25Y 0.6: 10Y)</th><th>Average maturity</th><th>Diff. in %</th><th>2 maturities (0.4: 1.25Y 0.6: 10Y)</th><th>Average maturity</th><th>Diff. in %</th></tr>
<tr><td>Short-term debt</td><td>40.17 (1.00)</td><td rowspan="2">102.47 (1.02)</td><td>+2</td><td>40.16 (1.00)</td><td rowspan="2">102.38 (1.02)</td><td>+2</td><td>40.18 (1.00)</td><td rowspan="2">103.31 (1.03)</td><td>+3</td></tr>
<tr><td>Long-term debt</td><td>63.06 (1.05)</td><td>-3</td><td>63.68 (1.06)</td><td>-4</td><td>63.21 (1.05)</td><td>-2</td></tr>
<tr><td>Firm value</td><td>229.17</td><td>231.26</td><td>+1</td><td>215.62</td><td>217.16</td><td>+1</td><td>230.16</td><td>234.53</td><td>+2</td></tr>
<tr><td>Equity value</td><td>125.93</td><td>128.79</td><td>+2</td><td>111.78</td><td>114.79</td><td>+3</td><td>126.77</td><td>131.21</td><td>+4</td></tr>
</table>

Table D1. Two maturity classes versus one: This table displays the main model outputs for the endogenous, the liquidity constraint, and the KMV-type default barrier under the base case scenario. Values in columns headed "Average maturity" are based on applying Leland´s (1994b) one-maturity model to the weighted average debt maturity and the base case parameters of this paper. The numbers in brackets are market values per unit (i.e., market value / nominal value).

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